Chapter 1: Limits and Continuity

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Limits are the foundation upon which all of calculus is built. Before you can understand derivatives or integrals, you need a solid grasp of what it means for a function to *approach* a value. In this chapter, we develop the concept of a limit from intuition to formal definition, explore techniques for evaluating limits, and then use limits to define what it means for a function to be continuous.

🎯 Learning Objectives

After completing this chapter, you will be able to:

- State the informal and formal definition of a limit and evaluate limits numerically
- Compute one-sided limits and determine whether a two-sided limit exists
- Evaluate limits using direct substitution, factoring, and rationalization
- Find limits at infinity and identify horizontal and vertical asymptotes
- Apply the Squeeze Theorem to evaluate limits involving oscillating functions
- Test continuity at a point, classify types of discontinuity, and apply the Intermediate Value Theorem

Chapter Contents

1.1 Convergence, Divergence & the Language of Limits

Before defining limits of functions formally, we build the vocabulary through **sequences** — infinite lists of numbers. Two facts are so fundamental they can be treated as common sense, and virtually every other limit result in this chapter reduces to them:

The Two Foundational Limits

$$\lim_{n\to\infty} n = \infty \qquad \text{(grows without bound — diverges)}$$

$$\lim_{n\to\infty} \frac{1}{n} = 0 \qquad \text{(shrinks to nothing — converges to 0)}$$

These are the *atoms* of limit reasoning. Every technique in Sections 1.3–1.5 ultimately reduces to one of these two patterns.

Think of them as common sense: if you keep counting ($n = 1, 2, 3, \ldots$), the numbers grow forever. If you take the reciprocal ($1, \frac{1}{2}, \frac{1}{3}, \ldots$), they shrink toward zero. We will use these two facts constantly to build every other limit result.

Two Types of Sequence Behavior

Definition: Convergence and Divergence

A **sequence** $\{a_n\}$ is an ordered list $a_1, a_2, a_3, \ldots$ indexed by positive integers. As $n \to \infty$, every sequence either **converges** or **diverges** — determined by whether a limit value exists:

Converges — Limit *EXISTS*

Terms approach a fixed value L:

$$\lim_{n\to\infty} a_n = L$$

Diverges — Limit *DNE*

Two sub-types:

- **Diverges to $\pm\infty$** — terms grow without bound
- **Oscillates** — terms alternate without settling

Sequence $\{a_n\}$	First few terms	Behavior	$\lim_{n\to\infty} a_n$
$\{n\}$	$1,\ 2,\ 3,\ 4,\ \ldots$	Diverges to $+\infty$	∞
$\{1/n\}$	$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$	Converges	0
$\left\{\frac{n}{n+1}\right\}$	$\frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \ldots$	Converges	1
$\{(-1)^n\}$	$-1,\ 1,\ -1,\ 1,\ \ldots$	Diverges (oscillates)	**DNE**
$\{n^2\}$	$1,\ 4,\ 9,\ 16,\ \ldots$	Diverges to $+\infty$	∞

Figure 1.1a — Convergence. $\{1/n\} \to 0$ and $\{n/(n+1)\} \to 1$. Limit **exists.**

Figure 1.1b — Divergence to $+\infty$. $\{n\}$ grows without bound. Limit **DNE**.

Figure 1.1c — Oscillation (a sub-type of divergence). $\{(-1)^n\}$ bounces between ± 1. Limit **DNE.**

Indeterminate vs. Determinate Forms

When you substitute a value into a limit expression and get an ambiguous result, you have an **indeterminate form**. The word "indeterminate" means: from the expression alone, you *cannot* determine the limit's value — the true answer could be any number, or DNE.

Definition: Indeterminate and Determinate Forms

Indeterminate forms — further algebraic or calculus work is required:

$$\frac{0}{0} \quad \frac{\infty}{\infty} \quad 0 \cdot \infty \quad \infty - \infty \quad 1^{\infty} \quad 0^0 \quad \infty^0$$

Determinate forms — the limit value is clear and unambiguous:

$\frac{c}{\infty} = 0$	finite divided by infinity
$c + \infty = \infty$	finite plus infinity
$c \cdot \infty = \infty$	positive finite times infinity
$\frac{c}{0^+} = +\infty$	positive constant over positive infinitesimal

★ **AP Exam Tip:** When substitution gives $\frac{0}{0}$, **do not** conclude the limit is 0, 1, or undefined. This is an indeterminate form — the actual limit could be anything. Proceed to factor, rationalize, or apply another technique from Section 1.3.

Worked Example 1.1 — Classifying Sequence Behavior

Classify each sequence as convergent, divergent, or oscillating, and state the limit (or DNE).

a. $\lim_{n\to\infty} \frac{5}{n}$

b. $\lim_{n\to\infty} \frac{n^2+1}{n^2}$

c. $\lim_{n\to\infty} (-1)^n$

Solutions

a. $\frac{5}{n} = 5 \cdot \frac{1}{n} \to 5 \cdot 0 = 0.$ **Converges to 0.** (c/∞ — determinate.)

b. $\frac{n^2+1}{n^2} = 1 + \frac{1}{n^2} \to 1 + 0 = 1.$ **Converges to 1.**

c. Terms: $-1, +1, -1, +1, \ldots$ **Oscillates — DNE.**

TRY IT

Classify the form of each limit, then find the limit if it exists.

a. $\lim_{n\to\infty} \frac{3n+1}{n}$

b. $\lim_{n\to\infty} \frac{7}{n^3}$

Limit Calculation Properties

The examples above used intuition: we could *see* that $5/n$ shrinks toward zero, or that $(-1)^n$ flips back and forth. But most sequences you will encounter in calculus are **not** so transparent. Consider

$$\lim_{n\to\infty} \frac{n^3+n^2+1}{n^3+1}, \qquad \lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right), \qquad \lim_{n\to\infty} \frac{1+2+\cdots+n}{n^2}.$$

None of these can be read off by inspection — substituting $n = \infty$ directly produces an indeterminate form. To move beyond intuition and *calculate* limits rigorously, we rely on a small set of arithmetic rules that hold whenever the individual limits exist and are finite.

Theorem: Limit Calculation Properties

Let $\{a_n\}$ and $\{b_n\}$ be sequences with $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, where L and M are **finite** real numbers. Then:

Property	Rule	In words
Sum	$\lim(a_n + b_n) = L + M$	Limit of a sum = sum of limits
Difference	$\lim(a_n - b_n) = L - M$	Limit of a difference = difference of limits
Scalar multiple	$\lim(c \cdot a_n) = c \cdot L$	Constants factor out of limits
Product	$\lim(a_n \cdot b_n) = L \cdot M$	Limit of a product = product of limits
Quotient	$\lim \frac{a_n}{b_n} = \frac{L}{M}$ $(M \neq 0)$	Limit of a quotient = quotient of limits *(denominator limit nonzero)*
Power	$\lim(a_n)^p = L^p$	Exponent passes through the limit
Root	$\lim \sqrt[p]{a_n} = \sqrt[p]{L}$ $(L \geq 0)$	Radical passes through the limit

Critical condition: All properties require both limits L and M to be finite. Arithmetic on ∞ is not covered by these rules — that is exactly when an indeterminate form arises and additional technique is needed.

★ **Why these matter:** These rules are the engine behind every algebraic manipulation in this chapter. When you "divide by n^2" or "multiply by the conjugate," you are applying combinations of these properties step by step — breaking an indeterminate limit into a chain of simpler limits, each of which has a finite, knowable value.

Worked Example 1.2 — Using Properties Step by Step

Find $\lim_{n\to\infty}\left(\frac{3}{n}+2\right)^2$ by citing limit properties explicitly.

Step 1 — Scalar + foundational limit. $\lim_{n\to\infty}\frac{3}{n}=3\cdot\lim_{n\to\infty}\frac{1}{n}=3\cdot 0=0$ (scalar multiple, then $1/n\to 0$).

Step 2 — Sum. $\lim_{n\to\infty}\left(\frac{3}{n}+2\right)=0+2=2$ (sum property; $\lim 2=2$ since constants are fixed).

Step 3 — Power. $\lim_{n\to\infty}\left(\frac{3}{n}+2\right)^2=2^2=\boxed{4}$ (power property with $p=2$).

Notice that at each step, the limit we were using was *finite*, so each property applied legally.

Six Standard Techniques for Sequence Limits

Armed with the calculation properties, we can now tackle the indeterminate cases. The core strategy is always the same: **transform the expression algebraically until each piece has a finite limit, then apply the properties above.** The following six worked examples each demonstrate one key transformation technique.

Worked Example 1.3 — Divide by the Highest Power (Finite Limit)

Find $\lim_{n\to\infty} \dfrac{n^3+n^2+1}{n^3+1}$.

Step 1 — Identify the form. Both numerator and denominator grow like n^3, so the raw form is $\dfrac{\infty}{\infty}$ — indeterminate.

Step 2 — Divide every term by n^3, the highest power present in the expression:

$$\frac{n^3+n^2+1}{n^3+1} = \frac{1+\dfrac{1}{n}+\dfrac{1}{n^3}}{1+\dfrac{1}{n^3}}$$

Step 3 — Take the limit. Every term with n in the denominator satisfies $\frac{1}{n^k} \to 0$:

$$\longrightarrow \frac{1+0+0}{1+0} = \boxed{1}$$

Rule of thumb: Same leading degree in numerator and denominator $\Rightarrow$ limit equals the ratio of the leading coefficients.

Worked Example 1.4 — Divide by the Highest Power (Limit = 0)

Find $\lim_{n\to\infty} \dfrac{n}{n^2+1}$.

Step 1 — Identify the form. Numerator has degree 1, denominator degree 2. Raw form: $\dfrac{\infty}{\infty}$.

Step 2 — Divide every term by n^2 (the higher degree):

$$\frac{n}{n^2+1} = \frac{\frac{1}{n}}{1+\frac{1}{n^2}}$$

Step 3 — Take the limit:

$$\longrightarrow \frac{0}{1+0} = \boxed{0}$$

Rule of thumb: Numerator degree < denominator degree $\Rightarrow$ limit $= 0$. The denominator "wins the race" to infinity.

Worked Example 1.5 — Numerator Degree > Denominator (Diverges to ∞)

Find $\lim_{n\to\infty} \frac{n^2}{n+1}$.

Step 1 — Identify the form. Numerator degree 2, denominator degree 1. Raw form: $\frac{\infty}{\infty}$.

Step 2 — Divide every term by n (the highest degree in the denominator):

$$\frac{n^2}{n+1} = \frac{n}{1+\frac{1}{n}}$$

Step 3 — Take the limit. The numerator $\to \infty$, denominator $\to 1$:

$$\longrightarrow \frac{\infty}{1} = \boxed{\infty} \quad (\text{diverges})$$

Rule of thumb: Numerator degree > denominator degree $\Rightarrow$ the sequence diverges to $\pm\infty$. The numerator "wins."

Worked Example 1.6 — Factor out the Highest Power ($\infty - \infty$ Form)

Find $\lim_{n\to\infty}(n^2 - n^3)$.

Step 1 — Identify the form. Both terms blow up as $n \to \infty$: $\infty - \infty$ — indeterminate. *Never* try to subtract or cancel infinities directly.

Step 2 — Factor out the highest power of n, which is n^3:

$$n^2 - n^3 = n^3\left(\frac{1}{n} - 1\right)$$

Step 3 — Evaluate each factor separately as $n \to \infty$:

- $n^3 \longrightarrow +\infty$
- $\frac{1}{n} - 1 \longrightarrow 0 - 1 = -1$

$$n^3 \cdot (-1) \longrightarrow \boxed{-\infty} \quad (\text{diverges to } -\infty)$$

Rule of thumb: Factor out the dominant power; the sign of the remaining bracket determines whether the sequence diverges to $+\infty$ or $-\infty$.

Worked Example 1.7 — Rationalize Using the Conjugate ($\infty - \infty$ with Roots)

Find $\lim_{n\to\infty}\left(\sqrt{n^2+1}-n\right)$.

Step 1 — Identify the form. For large n, $\sqrt{n^2+1}\approx n$, so this is $\infty - \infty$ — indeterminate. Factoring alone won't help because of the square root.

Step 2 — Multiply numerator and denominator by the conjugate $\left(\sqrt{n^2+1}+n\right)$:

$$\left(\sqrt{n^2+1}-n\right)\cdot\frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n}=\frac{(n^2+1)-n^2}{\sqrt{n^2+1}+n}=\frac{1}{\sqrt{n^2+1}+n}$$

Step 3 — Take the limit. The denominator grows without bound:

$$\frac{1}{\sqrt{n^2+1}+n}\longrightarrow\frac{1}{\infty}=\boxed{0}$$

Rule of thumb: Whenever you see $\sqrt{\text{polynomial}}-\text{polynomial}$, multiply by the conjugate. The difference of squares in the numerator collapses the indeterminate form.

Worked Example 1.8 — Apply a Summation Formula

Find $\lim_{n\to\infty} \dfrac{1+2+3+\cdots+n}{n^2}$.

Step 1 — Identify the form. The numerator grows with n; raw form is $\dfrac{\infty}{\infty}$.

Step 2 — Replace the sum with its closed-form formula. The arithmetic series identity gives:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Step 3 — Substitute and simplify:

$$\frac{n(n+1)/2}{n^2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$$

Step 4 — Take the limit:

$$\frac{1}{2} + \frac{1}{2n} \longrightarrow \frac{1}{2} + 0 = \boxed{\frac{1}{2}}$$

Rule of thumb: Any expression written with "$\cdots$" must be converted to a closed-form expression before taking limits. Two formulas to memorize:

- $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

These appear again in Riemann sum problems in Chapter 4.

TRY IT

Apply the appropriate technique to each limit.

a. $\lim_{n\to\infty} \frac{2n^2+3n}{n^2-1}$

b. $\lim_{n\to\infty} \left(\sqrt{n^2+4n}-n\right)$

c. $\lim_{n\to\infty} \frac{1^2+2^2+\cdots+n^2}{n^3}$ *(use* $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$*)*

1.2 Limits of Functions

Consider the function $f(x) = \frac{x^2-1}{x-1}$. If you try to evaluate $f(1)$ directly, you get $\frac{0}{0}$, which is undefined. But what happens to the values of $f(x)$ as x gets closer and closer to 1?

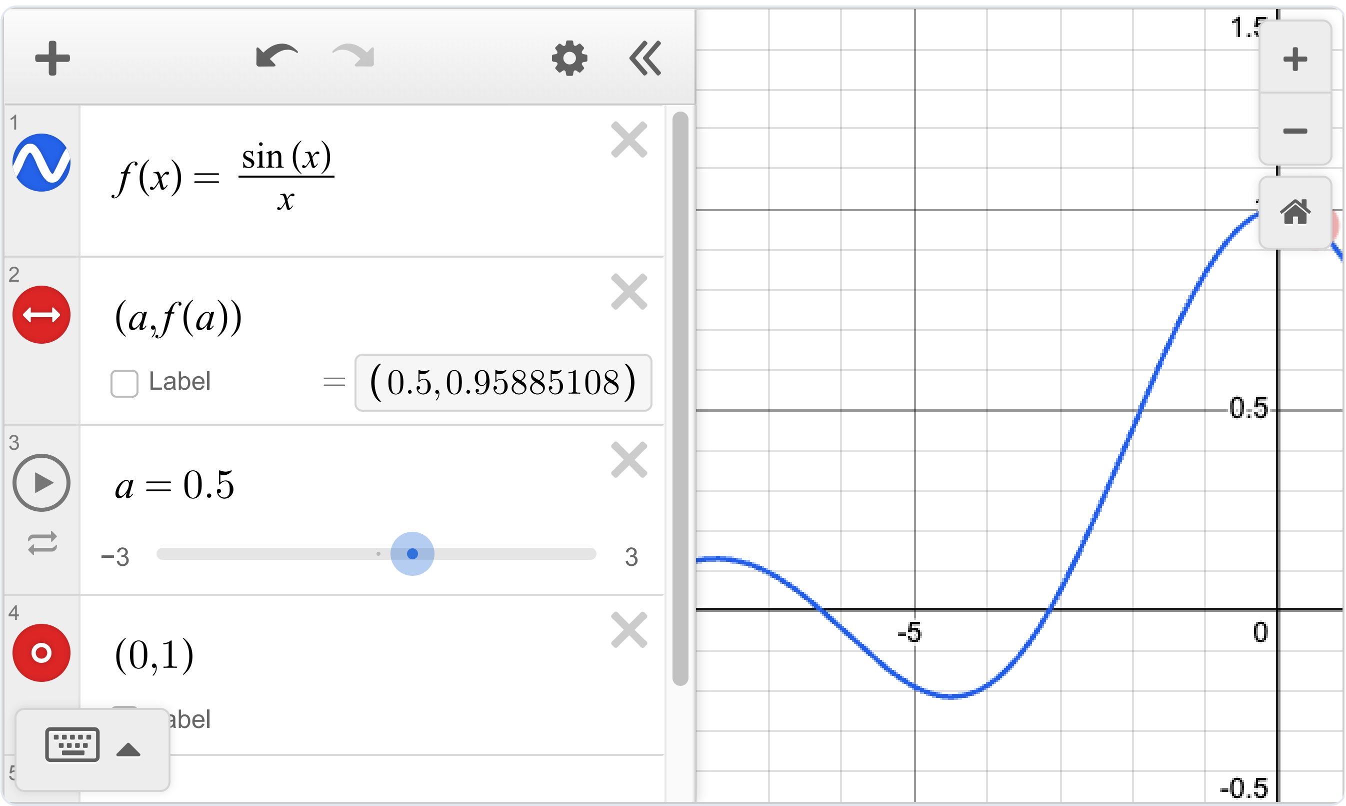
If we plug in values near 1, something interesting emerges:

- $f(0.9) = 1.9$
- $f(0.99) = 1.99$
- $f(0.999) = 1.999$
- $f(1.001) = 2.001$
- $f(1.01) = 2.01$
- $f(1.1) = 2.1$

Even though $f(1)$ is undefined, the output values clearly approach 2 as x approaches 1. This is the core idea of a limit: we care about what value the function is *heading toward*, not necessarily what value (if any) the function actually takes.

> **Definition: Limit of a Function**
>
> We write
>
> $$\lim_{x \to a} f(x) = L$$
>
> and say "the limit of $f(x)$ as x approaches a equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a (but not equal to a).

A crucial point: the limit depends on what happens *near* a, not *at* a. The function does not even need to be defined at $x = a$ for the limit to exist.



Explore limits: observe how $f(x) = \sin(x)/x$ approaches 1 as $x \to 0$, even though $f(0)$ is undefined. Drag the slider to move the point along the curve.

One-Sided Limits

Sometimes a function approaches different values depending on whether x approaches from the left or from the right. We distinguish between these with one-sided limits.

Definition: One-Sided Limits

Left-hand limit: $\lim_{x \to a^-} f(x) = L$ means $f(x) \to L$ as x approaches a from values *less than* a.

Right-hand limit: $\lim_{x \to a^+} f(x) = L$ means $f(x) \to L$ as x approaches a from values *greater than* a.

The two-sided limit $\lim_{x \to a} f(x) = L$ exists if and only if both one-sided limits exist and are equal:

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Worked Example 1.2 — One-Sided Limits

Consider the piecewise function

$$g(x) = \begin{cases} x + 1, & x < 2 \\ 5, & x = 2 \\ 7 - x, & x > 2 \end{cases}$$

Find $\lim_{x \to 2} g(x)$, if it exists.

Step 1 Find the left-hand limit. For $x < 2$, we use $g(x) = x + 1$:

$$\lim_{x \to 2^-} g(x) = \lim_{x \to 2^-} (x + 1) = 3$$

Step 2 Find the right-hand limit. For $x > 2$, we use $g(x) = 7 - x$:

$$\lim_{x \to 2^+} g(x) = \lim_{x \to 2^+} (7 - x) = 5$$

Step 3 Compare. Since $3 \neq 5$, the one-sided limits are not equal. Therefore, $\lim_{x \to 2} g(x)$ **does not exist**.

Note: $g(2) = 5$, but this is irrelevant to whether the limit exists. The limit fails because the function approaches different values from the left and right.

TRY IT

Consider $h(x) = \begin{cases} 3x - 1, & x < 2 \\ x^2 - 1, & x \geq 2 \end{cases}$. Does $\lim_{x \to 2} h(x)$ exist? If so, find it.

1.3 Evaluating Limits

Now that we understand what limits mean, we need reliable techniques for computing them. The AP Calculus exam tests several core methods, which we cover here in increasing order of sophistication.

Direct Substitution

The simplest and most powerful technique: if the function is continuous at $x = a$ (we will formalize this in Section 1.6), then

$$\lim_{x \to a} f(x) = f(a)$$

This works for all polynomials, rational functions (where the denominator is nonzero), trigonometric functions, exponentials, and logarithms at points in their domains. **Always try direct substitution first.**

★ **AP Exam Tip:** On the exam, if direct substitution gives a real number (not $\frac{0}{0}$ or $\frac{\infty}{\infty}$), that *is* the answer — no further work needed. Save time by trying substitution before anything else.

Worked Example 1.3a — Direct Substitution

Evaluate $\lim_{x \to 3} (2x^2 - 5x + 1)$.

Solution Since polynomials are continuous everywhere, we substitute directly:

$$\lim_{x \to 3} (2x^2 - 5x + 1) = 2(3)^2 - 5(3) + 1 = 18 - 15 + 1 = \boxed{4}$$

TRY IT

Evaluate $\lim_{x \to -2} (x^3 + 3x - 1)$.

Factoring and Canceling

When direct substitution yields the indeterminate form $\frac{0}{0}$, the numerator and denominator share a common factor of $(x - a)$. Factor it out, cancel, and then substitute.

Worked Example 1.3b — Factoring

Evaluate $\lim_{x \to 4} \frac{x^2 - 16}{x - 4}$.

Step 1 Try direct substitution: $\frac{16 - 16}{4 - 4} = \frac{0}{0}$. Indeterminate — we need another approach.

Step 2 Factor the numerator using the difference of squares:

$$\frac{x^2 - 16}{x - 4} = \frac{(x - 4)(x + 4)}{x - 4}$$

Step 3 Cancel the common factor (valid for $x \neq 4$):

$$= x + 4$$

Step 4 Now substitute:

$$\lim_{x \to 4} (x + 4) = \boxed{8}$$

TRY IT

Evaluate $\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$.

Rationalization (Conjugate Method)

When you encounter $\frac{0}{0}$ and the expression involves square roots, multiply by the **conjugate** of the expression containing the radical. The conjugate of $\sqrt{a} - b$ is $\sqrt{a} + b$.

Worked Example 1.3c — Rationalization

Evaluate $\lim_{x \to 0} \dfrac{\sqrt{x+9} - 3}{x}$.

Step 1 Direct substitution gives $\dfrac{0}{0}$. Indeterminate.

Step 2 Multiply by the conjugate of the numerator:

$$\frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \frac{(x+9)-9}{x(\sqrt{x+9}+3)} = \frac{x}{x(\sqrt{x+9}+3)}$$

Step 3 Cancel x (valid for $x \neq 0$) and substitute:

$$\lim_{x \to 0} \frac{1}{\sqrt{x+9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

TRY IT

Evaluate $\lim_{x \to 0} \dfrac{\sqrt{4+x} - 2}{x}$.

1.4 Limits Involving Infinity

So far we have considered limits where x approaches a finite number. We now extend the concept in two important ways: what happens as x grows without bound, and what happens when function values grow without bound.

Limits at Infinity and Horizontal Asymptotes

Definition: Limit at Infinity

We write $\lim_{x \to \infty} f(x) = L$ if the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. If this limit exists, then $y = L$ is a **horizontal asymptote** of the graph of f.

For rational functions $\frac{p(x)}{q(x)}$, the behavior as $x \to \infty$ follows a simple rule based on degrees:

- **deg(p) < deg(q):** the limit is 0.
- **deg(p) = deg(q):** the limit is the ratio of leading coefficients.
- **deg(p) > deg(q):** the limit is $\pm\infty$ (no horizontal asymptote).

Worked Example 1.4a — Horizontal Asymptote

Evaluate $\lim_{x \to \infty} \frac{5x^2 - 3x + 1}{2x^2 + 7}$.

Step 1 Both numerator and denominator have degree 2. Divide every term by x^2:

$$\frac{5 - \frac{3}{x} + \frac{1}{x^2}}{2 + \frac{7}{x^2}}$$

Step 2 As $x \to \infty$, all terms with x in the denominator approach 0:

$$\lim_{x \to \infty} \frac{5 - 0 + 0}{2 + 0} = \boxed{\frac{5}{2}}$$

Therefore $y = \frac{5}{2}$ is a horizontal asymptote.

TRY IT

Evaluate $\lim_{x \to \infty} \frac{3x^2 - 1}{x^3 + 2x}$.

Infinite Limits and Vertical Asymptotes

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of f if at least one of the following is true:

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Worked Example 1.4b — Vertical Asymptote

Find all vertical asymptotes of $f(x) = \dfrac{x+1}{x^2-4}$ and evaluate the one-sided limits at $x = 2$.

Step 1 Factor the denominator: $x^2 - 4 = (x-2)(x+2)$. Zeros at $x = 2$ and $x = -2$.

Step 2 Check the numerator at these points. At $x = 2$: numerator $= 3 \neq 0$. At $x = -2$: numerator $= -1 \neq 0$. Neither cancels, so **both are vertical asymptotes**.

Step 3 Analyze the sign near $x = 2$. The numerator $\to 3 > 0$ and $(x+2) \to 4 > 0$ on both sides.

- As $x \to 2^+$: $(x-2) \to 0^+$, so $f(x) \to +\infty$.
- As $x \to 2^-$: $(x-2) \to 0^-$, so $f(x) \to -\infty$.

TRY IT

Find the vertical asymptote(s) of $g(x) = \dfrac{x-1}{x^2-x-6}$. Are there any holes?

1.5 The Squeeze Theorem

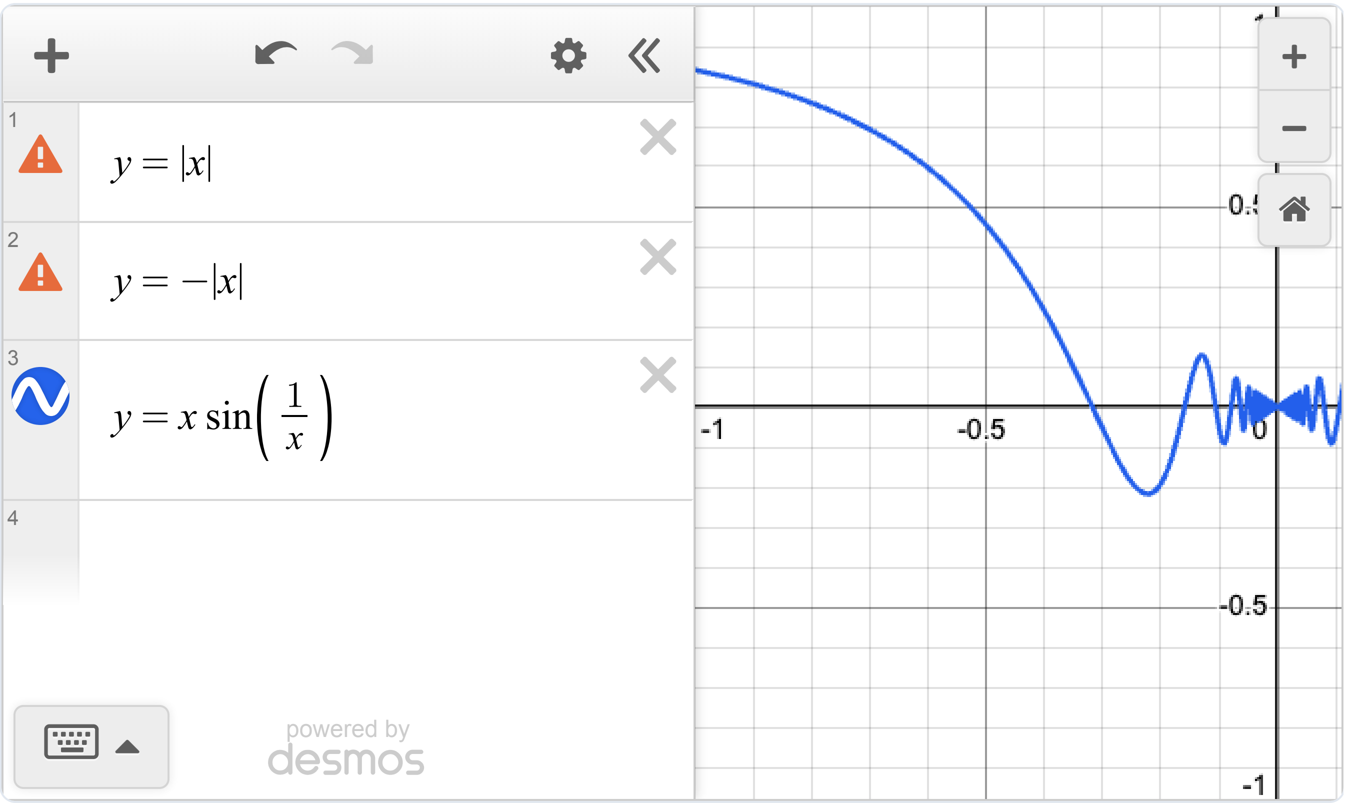
Some limits cannot be evaluated by direct substitution or algebraic manipulation. The Squeeze Theorem provides a way to evaluate limits by trapping the function between two simpler functions whose limits we already know.

Theorem: The Squeeze Theorem

Suppose $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing a (except possibly at a itself). If

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then $\lim_{x \to a} f(x) = L$ as well.



The Squeeze Theorem: $x \sin(1/x)$ (blue) is trapped between $-|x|$ and $|x|$ (red dashed), forcing the limit to 0 as $x \to 0$.

The Most Important Trigonometric Limits

Theorem 1.5 — Fundamental Trigonometric Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0 \qquad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

Proof of $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (Squeeze Theorem).

Assume $0 < x < \frac{\pi}{2}$. Consider a unit circle centered at the origin with a point $A = (1, 0)$, $B = (\cos x, \sin x)$ on the circle, and $T = (1, \tan x)$ where the tangent line meets the $x = 1$ line. Compare three areas:

- $\text{Area}(\triangle OAB) = \frac{1}{2}(1)(\sin x) = \frac{\sin x}{2}$
- $\text{Area}(\text{sector } OAB) = \frac{1}{2}(1)^2 x = \frac{x}{2}$
- $\text{Area}(\triangle OAT) = \frac{1}{2}(1)(\tan x) = \frac{\tan x}{2}$

Since triangle $\subseteq$ sector $\subseteq$ triangle:

$$\frac{\sin x}{2} \leq \frac{x}{2} \leq \frac{\tan x}{2}$$

Divide through by $\frac{\sin x}{2} > 0$:

$$1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

Take reciprocals (reversing inequalities):

$$\cos x \leq \frac{\sin x}{x} \leq 1$$

As $x \to 0^+$, $\cos x \to 1$ and $1 \to 1$. By the **Squeeze Theorem**:

$$\lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

For $x < 0$, write $x = -t$ with $t > 0$: $\frac{\sin x}{x} = \frac{\sin(-t)}{-t} = \frac{-\sin t}{-t} = \frac{\sin t}{t} \to 1$. Therefore $\lim_{x \to 0} \frac{\sin x}{x} = 1$. $\square$

Proof of $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$ (Rationalization).

Multiply numerator and denominator by the conjugate $1 + \cos x$:

$$\frac{1-\cos x}{x} = \frac{(1-\cos x)(1+\cos x)}{x(1+\cos x)} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)}$$

Rewrite by splitting the fraction:

$$= \frac{\sin x}{x} \cdot \frac{\sin x}{1+\cos x}$$

Taking the limit as $x \to 0$:

$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{\sin x}{1+\cos x} = 1 \cdot \frac{0}{1+1} = 1 \cdot 0 = 0 \qquad \square$$

Proof of $\lim_{x\to 0} \frac{\tan x}{x} = 1$.

Write $\tan x = \frac{\sin x}{\cos x}$ and split:

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{1} = 1 \qquad \square$$

Theorem 1.6 — General Trigonometric Limit Formulas

For constants $a \neq 0$ and b:

$$\lim_{x\to 0} \frac{\sin(bx)}{ax} = \frac{b}{a} \qquad \text{and} \qquad \lim_{x\to 0} \frac{\tan(bx)}{ax} = \frac{b}{a}$$

Proof. Let $u = bx$, so $u \to 0$ as $x \to 0$, and $x = u/b$:

$$\frac{\sin(bx)}{ax} = \frac{\sin u}{a \cdot (u/b)} = \frac{b}{a} \cdot \frac{\sin u}{u} \longrightarrow \frac{b}{a} \cdot 1 = \frac{b}{a}$$

The same substitution applies to $\tan(bx)/(ax)$. $\square$

★ **AP Exam Tip:** When you see $\frac{\sin(\text{something})}{x}$ or $\frac{\tan(\text{something})}{x}$, rewrite using the formula $\frac{b}{a}$ by matching the "something" to bx. For example: $\frac{\sin(3x)}{5x} = \frac{3}{5}$.

Worked Example 1.5 — Squeeze Theorem

Evaluate $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$.

Step 1 Direct substitution fails because $\cos(1/x)$ oscillates wildly near 0. However, we know $-1 \le \cos\left(\frac{1}{x}\right) \le 1$ for all $x \ne 0$.

Step 2 Multiply by $x^2 \ge 0$:

$$-x^2 \le x^2 \cos\left(\frac{1}{x}\right) \le x^2$$

Step 3 Both bounds $\to 0$ as $x \to 0$. By the Squeeze Theorem:

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = \boxed{0}$$

TRY IT

Evaluate $\lim_{x \to 0} \frac{\sin(5x)}{x}$.

1.6 Continuity

We now use limits to make precise the intuitive idea that a function is "continuous" if you can draw its graph without lifting your pen. But before we can state the definition of continuity, we need one essential tool: the ability to approach a point from a single direction.

One-Sided Limits

For most functions at most points, the limit is the same whether you approach from the left or the right. But at boundaries, corners, or abrupt changes, the two sides may behave very differently. The **one-sided limits** capture each side separately.

Definition: Left-Hand and Right-Hand Limits

The **left-hand limit** of f at $x = a$, written

$$\lim_{x \to a^-} f(x) = L,$$

is the value that $f(x)$ approaches as x approaches a from values *strictly less than* a (from the left side of a on the number line).

The **right-hand limit** of f at $x = a$, written

$$\lim_{x \to a^+} f(x) = M,$$

is the value that $f(x)$ approaches as x approaches a from values *strictly greater than* a (from the right side).

Key Theorem: The two-sided limit exists if and only if both one-sided limits exist *and agree*:

$$\lim_{x \to a} f(x) = L \quad \Longleftrightarrow \quad \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

If the left and right limits are different, the two-sided limit does not exist (DNE).

Figure 1.6a — One-sided limits. The blue curve approaches **3** from the left ($x \to 2^-$) and the green curve approaches **5** from the right ($x \to 2^+$). Because $3 \neq 5$, the two-sided limit at $x = 2$ does not exist.

Worked Example 1.6.0 — Evaluating One-Sided Limits

Let $f(x) = \begin{cases} x^2 - 1, & x < 2 \\ 5, & x = 2 \\ 2x - 1, & x > 2 \end{cases}$. Find $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$, $f(2)$, and $\lim_{x \to 2} f(x)$.

Left-hand limit. For $x < 2$, use the first piece $f(x) = x^2 - 1$:

$$\lim_{x \to 2^-} f(x) = (2)^2 - 1 = 3.$$

Right-hand limit. For $x > 2$, use the third piece $f(x) = 2x - 1$:

$$\lim_{x \to 2^+} f(x) = 2(2) - 1 = 3.$$

Function value. At $x = 2$ exactly, use the middle piece: $f(2) = 5$.

Two-sided limit. Since $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = 3$, the two-sided limit exists:

$$\lim_{x \to 2} f(x) = 3.$$

Conclusion: The limit is 3, but $f(2) = 5 \neq 3$. The two-sided limit exists but does not equal the function value — this is a *removable discontinuity*, as we will see in the next section.

TRY IT

For $g(x) = \begin{cases} 3x + 1, & x \leq 1 \\ x^2 + 3, & x > 1 \end{cases}$, find $\lim_{x \to 1^-} g(x)$, $\lim_{x \to 1^+} g(x)$, and $\lim_{x \to 1} g(x)$ if it exists.

Continuity at a Point

With one-sided limits in hand, we can state continuity precisely. A function is continuous at $x = a$ when three things happen simultaneously: the function is defined there, the limit exists there, and those two values agree.

Definition: Continuity at a Point

A function f is **continuous at** $x = a$ if all three of the following conditions hold:

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x \to a} f(x)$ exists (equivalently, $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$).
3. $\lim_{x \to a} f(x) = f(a)$.

If any one condition fails, f is **discontinuous** at $x = a$.

Types of Discontinuities

Not all discontinuities look alike. Depending on which condition fails — and how — we classify them into three types. Knowing the type tells you exactly what the graph looks like and whether the discontinuity can be "repaired."

Type 1: Removable Discontinuity (Hole)

Both one-sided limits agree — the two-sided limit $\lim_{x \to a} f(x) = L$ *exists* — but either:

- $f(a)$ is undefined (hole with no point), or
- $f(a)$ is defined but $f(a) \neq L$ (hole plugged at the wrong height).

It is called "removable" because redefining $f(a) = L$ **repairs** the discontinuity.

One-sided picture: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L, \quad \text{but} \quad f(a) \neq L$ (or undefined).

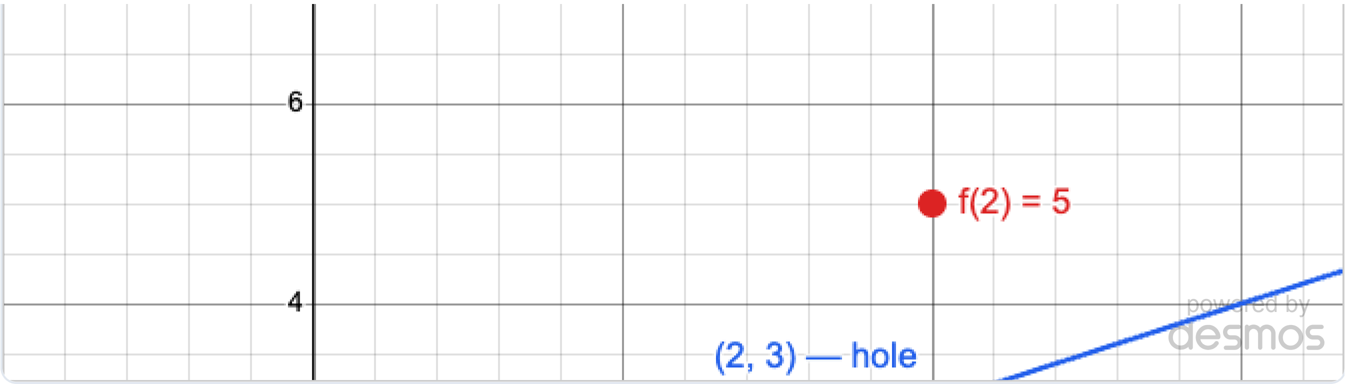


Figure 1.6b — Removable discontinuity at $x = 2$. Both the left and right limits equal 3 (open circle), but the function is defined at $f(2) = 5$ (filled red dot). The "hole" at $(2, 3)$ could be removed by redefining $f(2) = 3$.

Type 2: Jump Discontinuity

Both one-sided limits exist and are *finite*, but they are **unequal**:

$$\lim_{x \to a^-} f(x) = L \neq M = \lim_{x \to a^+} f(x).$$

The graph literally "jumps" from one height to another. The two-sided limit does not exist. Common in piecewise-defined functions and step functions. The value $f(a)$ may equal either one-sided limit, or neither.

Figure 1.6c — Jump discontinuity at $x = 2$. The left-hand limit is 2 (blue open circle) and the right-hand limit is 5 (green open circle). The graph "jumps" by 3 units. The filled dot shows $f(2) = 2$ (left value used).

Type 3: Essential (Infinite) Discontinuity

At least one one-sided limit is $\pm\infty$ — the function **blows up** near $x = a$. This creates a vertical asymptote on the graph. The discontinuity cannot be repaired by redefining $f(a)$.

$$\lim_{x \to a^-} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \pm\infty.$$

The word "essential" means the singularity is intrinsic to the function — there is no way to extend or redefine f to make it continuous at $x = a$.

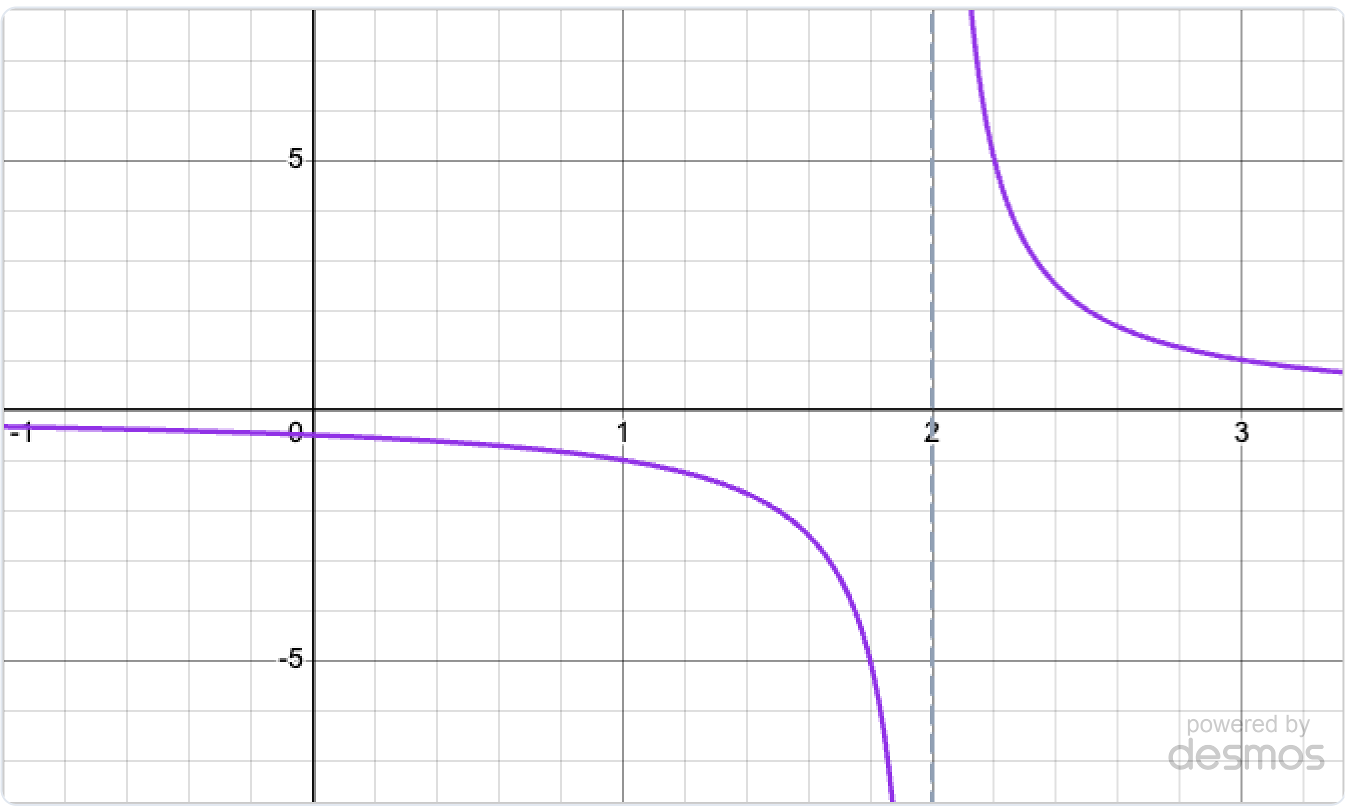


Figure 1.6d — Essential (infinite) discontinuity at $x = 2$ for $f(x) = \dfrac{1}{x-2}$. The left-hand limit is $-\infty$ and the right-hand limit is $+\infty$. The vertical dashed line $x = 2$ is the asymptote.

Type	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	Two-sided limit	Graph looks like	Repairable?
Removable	L	L	Exists ($= L$)	Single hole (open circle)	✓ Yes — redefine $f(a)$
Jump	L	$M \neq L$	DNE	Step / gap in graph	✗ No
Essential (infinite)	$\pm\infty$	$\pm\infty$	DNE	Vertical asymptote	✗ No

Worked Example 1.6a — Testing Continuity

Determine whether $f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & x \neq 3 \\ 7, & x = 3 \end{cases}$ is continuous at $x = 3$.

Step 1 Is $f(3)$ defined? Yes, $f(3) = 7$. ✓

Step 2 Does the limit exist?

$$\frac{x^2-9}{x-3} = \frac{(x-3)(x+3)}{x-3} = x+3 \quad (x \neq 3)$$

$$\lim_{x\to 3} f(x) = 6 \quad \checkmark$$

Step 3 Does limit = function value? We need $6 = 7$. **False. ✗**

Conclusion: f is **not continuous** at $x = 3$. This is a **removable discontinuity** — redefining $f(3) = 6$ would fix it.

TRY IT

Find the value of c so that $f(x) = \begin{cases} x^2+1, & x \leq 2 \\ cx-1, & x > 2 \end{cases}$ is continuous at $x = 2$.

✦ ENRICHMENT PREVIEW

The Intermediate Value Theorem

Full treatment coming in a later section — explore now if you feel ready, or skip and return later.

Skip for now ↓ continue

The Intermediate Value Theorem

Continuous functions cannot skip over values — they must pass through every intermediate value between their endpoints. This is one of the most tested theorems on the AP exam.

Theorem: Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there exists at least one value $c \in (a, b)$ such that $f(c) = N$.

★ **AP Exam Tip:** IVT questions always follow the same four-step structure: (1) define f, (2) verify f is continuous on $[a, b]$, (3) show N is between $f(a)$ and $f(b)$, (4) conclude by IVT. Always state "by the IVT" explicitly.

Worked Example 1.6b — Applying the IVT

Show that the equation $x^3 + x - 1 = 0$ has a solution in the interval $(0, 1)$.

Step 1 Define $f(x) = x^3 + x - 1$. As a polynomial, f is continuous everywhere, including on $[0, 1]$.

Step 2 Evaluate at endpoints: $f(0) = -1 < 0$ and $f(1) = 1 > 0$.

Step 3 Since $f(0) < 0 < f(1)$, the value $N = 0$ lies strictly between $f(0)$ and $f(1)$.

Step 4 By the IVT, there exists at least one $c \in (0, 1)$ such that $f(c) = 0$, i.e., $x^3 + x - 1 = 0$ has a solution in $(0, 1)$. $\square$

TRY IT

Show that $f(x) = e^x - 2$ has at least one zero on $(0, 1)$.

Worked Example 1.6c — Classifying All Discontinuities

Find all values of x where $f(x) = \dfrac{x^2 - x - 6}{x^2 - 4}$ is discontinuous, and classify each.

Step 1 Set the denominator to zero: $x^2 - 4 = 0 \Rightarrow x = \pm 2$. These are the only candidates.

Step 2 Factor the numerator: $x^2 - x - 6 = (x-3)(x+2)$.

Step 3 Check each candidate:

- **At $x = -2$:** Numerator $= (-2-3)(-2+2) = 0 \Rightarrow \frac{0}{0}$. Cancel the common factor $(x+2)$:

$$f(x) = \frac{(x-3)(x+2)}{(x-2)(x+2)} = \frac{x-3}{x-2} \quad (x \neq -2) \qquad \Rightarrow \qquad \lim_{x \to -2} f(x) = \frac{-5}{-4} = \frac{5}{4}$$

  Limit exists but $f(-2)$ is undefined: **removable discontinuity** at $x = -2$.

- **At $x = 2$:** Numerator $= (2-3)(2+2) = -4 \neq 0$. Denominator $\to 0$ while numerator $\to -4$, so $f(x) \to \pm\infty$: **infinite discontinuity** (vertical asymptote) at $x = 2$.

TRY IT

Find all discontinuities of $g(x) = \dfrac{x^2 - 5x + 6}{x^2 - 9}$ and classify each.

Worked Example 1.6d — IVT with a Transcendental Equation

Show that $2\sin x = x$ has at least one positive solution.

Step 1 Let $f(x) = 2\sin x - x$. Since $\sin x$ and x are continuous everywhere, so is f.

Step 2 Note $f(0) = 0$, so $x = 0$ is already a solution. For a *positive* solution, use the interval $\left[\frac{\pi}{2}, \pi\right]$:

$$f\left(\tfrac{\pi}{2}\right) = 2(1) - \tfrac{\pi}{2} = 2 - \tfrac{\pi}{2} \approx 0.43 > 0 \qquad f(\pi) = 2(0) - \pi = -\pi \approx -3.14 < 0$$

Step 3 Since $f\left(\tfrac{\pi}{2}\right) > 0 > f(\pi)$, the target value $N = 0$ lies strictly between the endpoint values.

Step 4 By the IVT, $\exists c \in \left(\frac{\pi}{2}, \pi\right)$ with $f(c) = 0$, i.e., $2\sin c = c$. Since $c > 0$, this is a positive solution. $\square$

TRY IT

Show that $\cos x = x$ has a solution in $\left(0, \frac{\pi}{2}\right)$.

1.7 Practice Problems

Basic Intermediate Challenge

Attempt each problem before revealing the solution. Problems are ordered from basic to challenging.

Problem 1 Basic

Evaluate $\lim_{x \to 5} \dfrac{x^2 - 25}{x - 5}$.

▶ Show Solution

Problem 2 Basic

Evaluate $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x}$.

▶ Show Solution

Problem 3 Basic

Evaluate $\lim_{x \to \infty} \frac{3x^3 - 2x}{7x^3 + 4x^2 - 1}$.

▶ Show Solution

Problem 4 Intermediate

Evaluate $\lim_{x \to 0} \frac{\sin(3x)}{x}$.

▶ Show Solution

Problem 5 Intermediate

Let $f(x) = \begin{cases} 2x+1, & x < 1 \\ x^2 + c, & x \geq 1 \end{cases}$. Find the value of c that makes f continuous at $x = 1$.

▶ Show Solution

Problem 6 Intermediate

Evaluate $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$.

▶ Show Solution

Problem 7 Intermediate

Use the Squeeze Theorem to evaluate $\lim_{x \to \infty} \frac{\sin x}{x}$.

▶ Show Solution

Problem 8 Challenge

Show that the equation $e^x = 3 - x$ has at least one solution on the interval $(0, 1)$.

▶ Show Solution

Problem 9 Challenge

Evaluate $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$.

▶ Show Solution

Problem 10 Challenge

Find all vertical and horizontal asymptotes of $f(x) = \frac{2x^2 + 3}{x^2 - 1}$.

▶ Show Solution

Problem 11 Basic

The graph below shows the function f. Use it to answer parts (a)–(d).

Figure 1.7 — Graph of f. Open circle at $(2,\ 1)$; filled circle at $(2,\ 3)$.

a. Find $\lim_{x \to 2^-} f(x)$.

b. Find $\lim_{x \to 2^+} f(x)$.

c. Find $\lim_{x \to 2} f(x)$, or state DNE.

d. Is f continuous at $x = 2$? Classify any discontinuity.

▶ Show Solution

Problem 12 Basic

Evaluate $\lim_{x \to 0} \dfrac{\sin(4x)}{2x}$.

▶ Show Solution

Problem 13 Basic

Let $f(x) = \begin{cases} 2x + 3, & x < 1 \\ 5, & x = 1 \\ x^2 + 1, & x > 1 \end{cases}$. Find the one-sided limits at $x = 1$, the two-sided limit, and state whether f is continuous at $x = 1$.

▶ Show Solution

Problem 14 Intermediate

The graph below shows $h(x)$. Identify and justify the type of discontinuity at each labeled point.

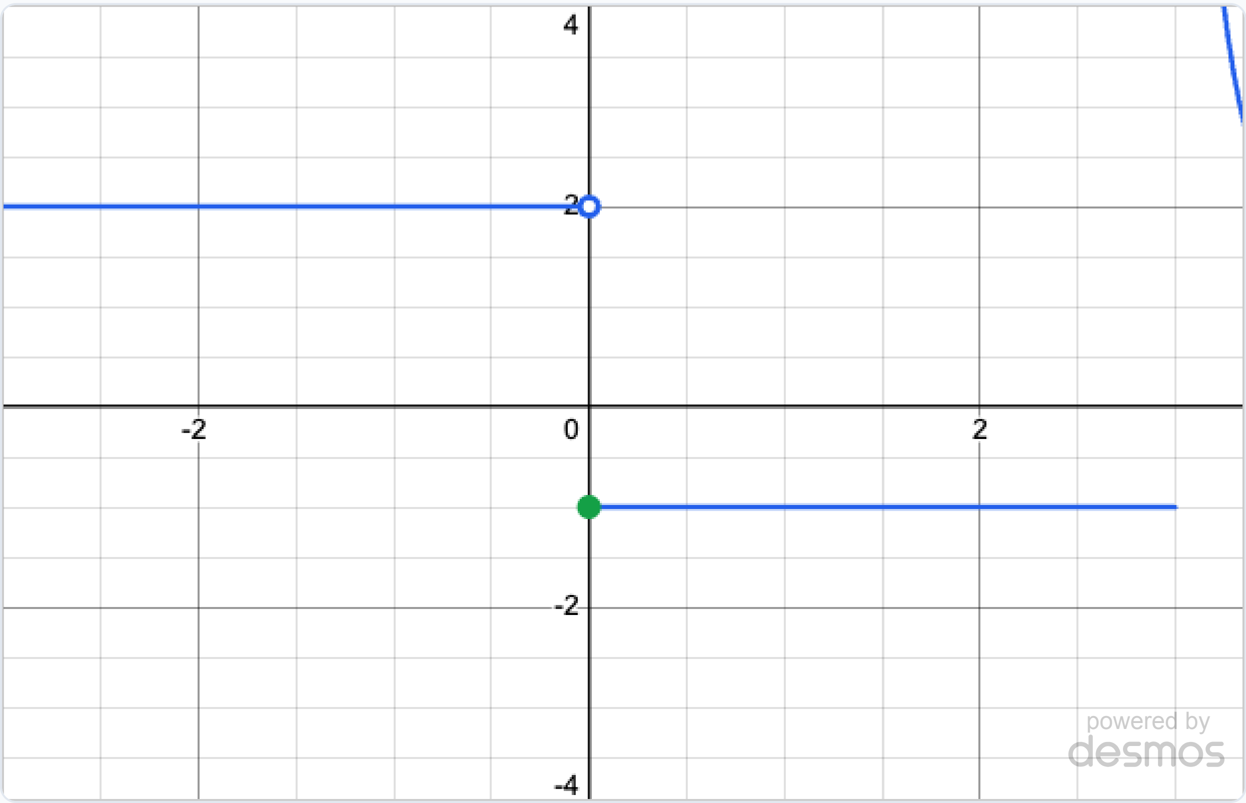


Figure 1.8 — Graph of $h(x)$. Open circle at $(0, 2)$; filled circle at $(0, -1)$; vertical asymptote at $x = 3$.

a. What type of discontinuity at $x = 0$?

b. What type of discontinuity at $x = 3$?

▶ Show Solution

Problem 15 Intermediate

Evaluate $\lim_{x \to 0} \dfrac{\sin(3x)}{\tan(2x)}$.

▶ Show Solution

Problem 16 Intermediate

On what set is $f(x) = \sqrt{x^2 - 9}$ continuous?

▶ Show Solution

Problem 17 Intermediate

Evaluate $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$.

▶ Show Solution

Problem 18 Intermediate

Find a and b so that f is continuous everywhere:

$$f(x) = \begin{cases} 3x + a, & x < -1 \\ bx^2 - 1, & -1 \leq x \leq 2 \\ x + 5, & x > 2 \end{cases}$$

▶ Show Solution

Problem 19 Intermediate

Evaluate $\lim_{x \to \infty} \frac{x}{\sqrt{x^2 + 1}}$.

▶ Show Solution

Problem 20 Intermediate

Use the IVT to show that $x^5 - 2x - 1 = 0$ has at least one solution on $(1, 2)$.

▶ Show Solution

Problem 21 Intermediate

Evaluate $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$.

▶ Show Solution

Problem 22 Intermediate

Evaluate $\lim_{x \to 3^-} \frac{|x-3|}{x-3}$, or state DNE.

▶ Show Solution

Problem 23 Challenge

The graph below shows a continuous function k on $[0, 4]$, with $k(0) = 5$ and $k(4) = 5$. A student claims the IVT guarantees $k(c) = 0$ for some $c \in (0, 4)$. Is the student correct? Explain fully.

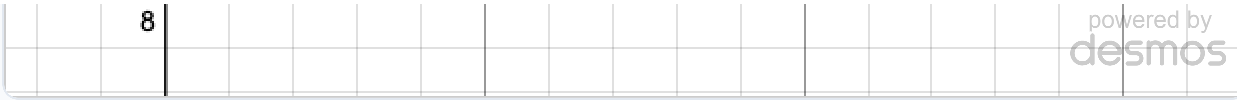


Figure 1.9 — Continuous function $k(x) = x^2 - 4x + 5$ on $[0, 4]$. Green dots: endpoints. Purple dot: minimum at $(2, 1)$. Red dashed line: $y = 0$.

▶ Show Solution

Problem 24 Challenge

Evaluate $\lim_{x \to 0} \dfrac{\sin^2 x}{x \tan x}$.

▶ Show Solution

Problem 25 Challenge

Evaluate $\lim_{h \to 0} \dfrac{\sqrt{9+h} - 3}{h}$.

▶ Show Solution

Problem 26 Challenge

Find a and b so that g is continuous on $(-\infty, \infty)$:

$$g(x) = \begin{cases} ax - 2, & x < 1 \\ b, & x = 1 \\ 3x^2 - a, & x > 1 \end{cases}$$

▶ Show Solution

Problem 27 Challenge

Evaluate $\lim_{x \to \pi} \dfrac{\sin x}{x - \pi}$.

▶ Show Solution

Problem 28 Challenge

Evaluate $\lim_{h \to 0} \dfrac{\sin\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}\right)}{h}$. *(Hint: expand with the sine addition formula.)*

▶ Show Solution

Problem 29 Challenge

Show that $f(x) = x^2 - \cos x$ has at least one zero in $(-1, 0)$.

▶ Show Solution

Problem 30 Challenge

Prove that every polynomial of odd degree has at least one real root. *(Hint: consider the end behavior as $x \to \pm\infty$.)*

▶ Show Solution

📋 Chapter Summary

CORE DEFINITIONS

Limit

$\lim_{x \to a} f(x) = L$ means $f(x)$ can be made arbitrarily close to L by taking x close to a (but $\neq a$).

Two-sided limit exists when

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Continuity at $x = a$

$f(a)$ defined, limit exists, and $\lim_{x \to a} f(x) = f(a)$.

Horizontal asymptote rule

Compare degrees of numerator and denominator. Equal degrees → ratio of leading coefficients.

KEY FORMULAS

Trigonometric limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Squeeze Theorem

$g \leq f \leq h$ and $\lim g = \lim h = L \Rightarrow \lim f = L$

EVALUATION STRATEGIES (IN ORDER)

1. **Direct substitution** — always try first
2. **Factor and cancel** — when you get $\frac{0}{0}$ and no radicals
3. **Rationalize** — when $\frac{0}{0}$ involves square roots
4. **Squeeze Theorem** — when the function oscillates
5. **Divide by highest power** — for limits at infinity

📘 Key Terms

Limit

The value a function approaches as the input approaches some point, regardless of what happens at that point.

One-sided limit

A limit where x approaches from only one direction: left (a^-) or right (a^+).

Indeterminate form

A form such as $\frac{0}{0}$ or $\frac{\infty}{\infty}$ that does not directly determine the limit's value.

Horizontal asymptote

A line $y = L$ that the function approaches as $x \to \pm\infty$.

Vertical asymptote

A line $x = a$ where the function grows without bound.

Squeeze Theorem

If a function is trapped between two functions that converge to L, it must also converge to L.

Continuity

A function is continuous at a when $f(a)$ is defined, the limit exists, and they are equal.

Removable discontinuity

A "hole" where the limit exists but the function value is missing or wrong. Can be fixed by redefining $f(a)$.

Jump discontinuity

The left and right limits both exist but are unequal, creating a "jump" in the graph.

Intermediate Value Theorem

A continuous function on $[a, b]$ takes on every value between $f(a)$ and $f(b)$. Used to prove existence of solutions.

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